Final Exam — Analysis (WPMA14004)

Thursday 16 June 2016, 9.00h–12.00h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (3 + 12 points)

(a) State the Axiom of Completeness.

(b) Assume that the sets $A, B \subset \mathbb{R}$ are both bounded above. Prove that

$$\sup(A \cup B) = \max\{\sup A, \sup B\}.$$

Hint: first explain that it suffices to consider only the case $\sup A \leq \sup B$.

Problem 2 (4 + 4 + 7 points)

Consider the sequences (t_k) and (s_n) given by

$$t_k = \frac{1}{k} - \ln\left(\frac{k+1}{k}\right) \qquad \text{and} \qquad s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1).$$

Prove the following statements:

(a) $\displaystyle\sum_{k=1}^{n} t_k = s_n$ for all $n \in \mathbb{N}$.

(b) $0 \leq t_k \leq \dfrac{1}{2k^2}$ for all $k \in \mathbb{N}$. Hint: $x - \frac{1}{2}x^2 \leq \ln(1+x) \leq x$ for all $x \geq 0$.

(c) (s_n) is convergent.

Problem 3 (5 + 10 points)

Let $B \subset \mathbb{R}$ be a set of positive real numbers with the following "finite sum property": adding finitely many elements of B gives a sum of 1 or less.

Prove the following statements:

(a) For all $\epsilon > 0$ there exist only finitely many $x \in B$ with $x > \epsilon$.

(b) $B \cup \{0\}$ is compact.

Problem 4 (4 + 4 + 7 points)

Consider the following function:

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \frac{x}{1+|x|}.$$

Prove the following statements:

(a) f is differentiable at $x = 0$ and $f'(0) = 1$.

(b) f is differentiable at $x \neq 0$ and $0 < f'(x) < 1$.

(c) f is uniformly continuous on $\mathbb{R}$.

Problem 5 (3 + 6 + 6 points)

Let $g : \mathbb{R} \to \mathbb{R}$ be a function with domain $\mathbb{R}$. Consider the following sequence:

$$f_n(x) = \frac{ng(x)}{n + |g(x)|}.$$

Prove the following statements:

(a) $|f_n(x) - g(x)| \leq \dfrac{g(x)^2}{n}$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

(b) If g is bounded on $\mathbb{R}$, then $f_n \to g$ uniformly on $\mathbb{R}$.

(c) If g is continuous on $\mathbb{R}$, then $f_n \to g$ uniformly on all compact subsets of $\mathbb{R}$.

Problem 6 (9 + 6 points)

Consider the modified Dirichlet function $h : [0, 1] \to \mathbb{R}$ defined by

$$h(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Show that $U(h, P) > \frac{1}{2}$ for any partition P of $[0, 1]$.

Hint: prove that $x_k(x_k - x_{k-1}) > \frac{1}{2}(x_k + x_{k-1})(x_k - x_{k-1})$.

(b) Is h integrable on $[0, 1]$?

End of test (90 points)